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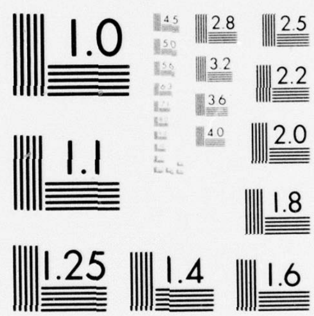

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NONPARAMETRIC TESTS FOR MULTIPLE REGRESSION UNDER PROGRESSIVE CENSORING*

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ABSTRACT

For continuous observations from time-sequential studies, suitable Cramér-von Mises and Kolmogorov-Smirnov type (nonparametric) statistics (based on linear rank statistics) for testing hypotheses on some multiple regression models are proposed and studied. Asymptotic theory of these tests is provided for both the null and (local) alternative hypotheses situations and is based on the weak convergence of suitable rank order processes (on the D[0,1] space) to certain functions of Brownian motions. Bahadur efficiency results are also presented. Empirical values of the percentile points of the null distributions of the proposed test statistics, obtained through simulation studies, are also provided.

AMS 1970 Classification Nos: 60F05, 62G10, 62L99

Key Words & Phrases: Bahadur-efficiency, clinical trials, Cramér-von statistics, D[0,1] space, Kolmogorov-Smirnov statistics, life testing, linear rank statistics, local (contiguous) alternatives, multiple regression, progressive censoring schemes, rank tests, time-sequential procedures, weak convergence, Wiener processes.

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1. INTRODUCTION

In longitudinal (time-sequential) studies relating to clincial trials and life testing problems, the experimenter sets out to plan beforehand as to the maximum number of responses to be observed or the maximum duration of the experimentation. As the experiment is continuously observed over time, even with this restricted design, the experimenter has the option of reviewing the outcome as the experiment progresses, enabling him to terminate the experiment at an intermediate stage if the cumulative evidence indicated by the data at that stage is strong enough to reject the null hypothesis and further continuation of experimentation can not lead to a different inference. This pseudo- sequential test procedure (which is distinguished from the classical sequential test) arises from what is called *progressive censoring schems* (PCS) as at the successive censoring time-points (responses or failures), the test statistics are based on uncensored subjects only. It may be mentioned that, for the applications of this procedure, it is not necessary that one works with a restricted design only.

For two-sample location and scale problems as well as the simple regression model, Chatterjee and Sen (1973) have developed a general class of (linear) rank statistics incorporated for testing under PCS. In the current investigation, their theory is extended to the multiple regression problem which includes the multi-sample location problem as a special case. Also, a wider class of test statistics is considered here.

Section 2 is devoted to the basic formulation of the problem. Section 3 deals with the development of PCS tests: two different types, viz., Kolomogorov-Smirnov and Cramér-von Mises, both based on linear rank statistics, are considered. Sections 4 and 5 are devoted to the study of the asymptotic distribution theory of the test statistics under the null and (local) contiguous alternative hypotheses. Asymptotic (Bahadur-) relative efficiency results are presented in Section 6. Section 7 deals with the simulation study of the null distributions of the proposed test statistics. In Section 8, comparison of the proposed PCS tests with the fixed-plan censoring modifications of the Kruskal-Wallis test by Basu (1967) has been made and the scope of applicability of the PCS tests in the context of right truncation is discussed.

2. PRELIMINARY NOTIONS

Let $\{X_i, i \geq 1\}$ be a sequence of independent random variables (r.v.) with continuous distribution functions (d.f.) $\{F_i, i \geq 1\}$, specified by the model

$$F_i(x) = F(x-\beta_0-\underset{\sim}{\beta}'\underset{\sim}{c}_i), \quad -\infty < x < \infty, \quad i \geq 1, \tag{2.1}$$

where in this conventional multiple regression model, the d.f. F is not known, β_0, $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_p)$ are unknown parameters, $p(\geq 1)$ and $\{\underset{\sim}{c}_i' = (c_{1i}, \ldots, c_{pi}), i \geq 1\}$ is a sequence of (known) vectors of regression constants. Our concern is to test

$$H_0: \underset{\sim}{\beta} = \underset{\sim}{0} \quad \text{vs.} \quad H_1: \underset{\sim}{\beta} \neq \underset{\sim}{0}, \tag{2.2}$$

(treating β_0 as a nuisance parameter and without assuming the form of F to be specified), when, in fact, we have a life testing model, as may be posed below.

For every $N(\geq 1)$, let $Z_{N1} \leq \dots \leq Z_{NN}$ be the ordered random variables corresponding to $(X_1, \dots, X_N) = \underset{\sim}{X}^{(N)}$, say; by virtue of the assumed continuity of the F_i, ties among the observations may be neglected, in probability. Let $\underset{\sim}{R}_N = (R_{N1}, \dots, R_{NN})'$ and $\underset{\sim}{S}_N = (S_{N1}, \dots, S_{NN})'$ be respectively the vectors of *ranks* and *anti-ranks* of the elements of $\underset{\sim}{X}^{(N)}$, so that (ties neglected), $R_{NS_{Ni}} = S_{NR_{Ni}} = i$, $1 \leq i \leq N$, $X_i = Z_{NR_{Ni}}$, $1 \leq i \leq N$. In a life testing problem, one typically observes the successive order statistics $\{Z_{Ni}\}$ along with the corresponding $\{S_{Ni}\}$, and based on a part of the sequence $\{Z_{Ni}, S_{Ni}; 1 \leq i \leq N\}$, the problem is to test for H_0 in (2.2). The PCS test is a pseudo-sequential procedure where a early termination of experimentation is feasible if observing $\{Z_{Ni}, S_{Ni}, 1 \leq i \leq k\}$ for some $k(\leq N)$, the accumulated statistical evidence leads to a decisive conclusion. Our proposed PCS tests are based on suitable linear rank statistics which we introduce in this section. The actual test statistics will be introduced in the next section.

For every $N(\geq 1)$, we conceive of a set of *scores* $\{a_N(1), \dots, a_N(N)\}$ generated by a *score function* $\phi = \{\phi(u), u \in (0,1)\}$ in the following way:

$$a_N([Nu] + 1) = \phi_N(u): 0 < u < 1 , \tag{2.3}$$

where $[s]$ denotes the largest integer $\leq s$ and

$$\lim_{N\to\infty} \int_0^1 \{\phi_N(u) - \phi(u)\}^2 du = 0 . \tag{2.4}$$

Further, $\phi(u)$ is assumed to be expressible as the difference of two $[\phi_1(u)$ and $\phi_2(u)]$ non-decreasing and square integrable (inside $I = [0,1]$) functions. Note that (2.4) holds, in particular, when

$$a_N(i) = E\phi(U_{Ni}) \quad \text{or} \quad \phi(i/(N+1)) \ , \quad 1 \le i \le N \ , \tag{2.5}$$

where $U_{N1} < \ldots < U_{NN}$ are the ordered random variables of a sample of size N from the rectangular $[0,1]$ d.f. In order to simplify the notations, in the sequal, we let

$$\bar{\phi} = \int_0^1 \phi(u)du = 0 \ , \quad A_\phi^2 = \int_0^1 \phi^2(u)du - \bar{\phi}^2 = \int_0^1 \phi^2(u)du = 1 \ , \tag{2.6}$$

$$\bar{a}_N = \frac{1}{N}\sum_{i=1}^N a_N(i) = 0 \quad \text{and} \quad A_N^2 = (N-1)^{-1}\sum_{i=1}^N [a_N(i) - \bar{a}_N]^2 = 1, \ N \ge 2 \ . \tag{2.7}$$

Now, at the k-th stage, the observable random variables are $\{Z_{Ni}, S_{Ni};\ 1 \le i \le k\}$, and based on these, we define

$$\underset{\sim}{T}_{N,k} = (T_{N,k}^{(1)}, \ldots, T_{N,k}^{(p)})', \quad (1 \le k \le N)$$

by letting

$$\begin{aligned} \underset{\sim}{T}_{N,k} &= \sum_{i=1}^k (\underset{\sim}{c}_{S_{Ni}} - \bar{\underset{\sim}{c}}_N) a_N(i) + \left[\sum_{i=k+1}^N (\underset{\sim}{c}_{S_{Ni}} - \bar{\underset{\sim}{c}}_N)\right] [(N-k)^{-1}\sum_{i=k+1}^N a_N(i)] \\ &= \sum_{i=1}^k (\underset{\sim}{c}_{S_{Ni}} - \bar{\underset{\sim}{c}}_N)[a_N(i) - a_N^*(k)] \ , \end{aligned} \tag{2.9}$$

where $\bar{\underset{\sim}{c}}_N = N^{-1}\sum_{i=1}^N \underset{\sim}{c}_i$ and

$$a_N^*(k) = \begin{cases} (N-k)^{-1}\sum_{i=k+1}^N a_N(i) \ , & 1 \le k \le N-1 \ , \\ 0 \ , & k = 0, \ N \ , \end{cases} \tag{2.10}$$

and, conventionally, we let $\underset{\sim}{T}_{N,0} = \underset{\sim}{0}$. Note that $\underset{\sim}{T}_{N,N-1} = \underset{\sim}{T}_{NN} = \underset{\sim}{T}_N$, where

$$\mathbf{T}_N = \sum_{i=1}^{N}(\mathbf{c}_{S_{Ni}} - \bar{\mathbf{c}}_N)a_N(i) = \sum_{i=1}^{N}(\mathbf{c}_i - \bar{\mathbf{c}}_N)a_N(R_{Ni}) \ . \tag{2.11}$$

Actually, if we let

$$b_{N,k}(i) = \begin{cases} a_N(i) \ , & 1 \le i \le k \ , \\ a_N^*(k) \ , & k < i \le N \ , \end{cases} \qquad k = 0,1,\ldots,N \ , \tag{2.12}$$

then, we may rewrite $\mathbf{T}_{N,k}$ as

$$\mathbf{T}_{N,k} = \sum_{i=1}^{N}(\mathbf{c}_i - \bar{\mathbf{c}}_N)b_{N,k}(R_{Ni}) \ , \quad k = 0,1,\ldots,N \ . \tag{2.13}$$

Note that by (2.7), (2.10) and (2.12), for every $1 \le k \le q \le N$,

$$\begin{aligned}(N-1)^{-1}\sum_{i=1}^{N} b_{N,k}(i)b_{N,q}(i) &= (N-1)^{-1}\left\{\sum_{i=1}^{k} a_N^2(i) + (N-k)[a_N^*(k)]^2\right\} \\ &= A_N^2 - (N-1)^{-1}\sum_{i=k+1}^{N}[a_N(i) - a_N^*(k)]^2 \qquad (2.14) \\ &= 1 - (N-1)^{-1}\sum_{i=k+1}^{N}[a_N(i) - a_N^*(k)]^2 = A_{N,k}^2 \ , \quad \text{say} \ ,\end{aligned}$$

where $A_{N,k}^2$ is $\nearrow$ in k $(0 \le k \le N)$ and $A_{N,0}^2 = 0$, $A_{N,N-1}^2 = A_{N,N}^2 = A_N^2 = 1$. Let us also denote

$$\mathbf{C}_N = \sum_{i=1}^{N}(\mathbf{c}_i - \bar{\mathbf{c}}_N)(\mathbf{c}_i - \bar{\mathbf{c}}_N)' = ((C_{Nj\ell}))_{j,\ell=1,\ldots,p} \ , \tag{2.15}$$

and assume that there exists a positive number N_0, such that

$$\mathbf{C}_N \text{ is positive-definite (p.d.)}, \ \forall\, N \ge N_0 \ . \tag{2.16}$$

We may remark that under H_0 in (2.2), $\mathbf{Z}_N = (Z_{N1},\ldots,Z_{NN})'$ and $\mathbf{R}_N$ (and $\mathbf{S}_N$) are stochastically independent; $\mathbf{R}_N$ (and $\mathbf{S}_N$) assumes all possible permutations of $(1,\ldots,N)$ with the equal probability $(N!)^{-1}$. Hence, from (2.13)-(2.15), we obtain by some routine computations that

$$E(\underset{\sim}{T}_{N,k} | H_0) = \underset{\sim}{0} \ , \quad 0 \le k \le N \ ; \tag{2.17}$$

$$E(\underset{\sim}{T}_{N,k} \underset{\sim}{T}'_{N,q} | H_0) = A^2_{N,k \wedge q} \cdot \underset{\sim}{C}_N \ , \quad 0 \le k \ , \quad q \le N \ , \tag{2.18}$$

where $a \wedge b = \min(a,b)$. Finally, we define

$$L_{N,k} = \begin{cases} 0 \ , & k = 0 \ , \\ (\underset{\sim}{T}'_{N,k} \underset{\sim}{C}^{-1}_N \underset{\sim}{T}_{N,k})^{\frac{1}{2}} \ , & k = 1, \dots, N \ . \end{cases} \tag{2.19}$$

Our proposed tests (under PCS) are based on the partial sequence

$$\{L_{N,k} : 0 \le k \le r\} \tag{2.20}$$

where r is any pre-assigned integer, such that

$$r/N \to \delta: \ 0 < \delta \le 1 \ , \quad \text{as} \quad N \to \infty \ . \tag{2.21}$$

In reality, mostly δ is less than one.

3. PCS RANK ORDER TESTS FOR NO REGRESSION

For every $N \ge r \ge 1$, we introduce a sequence $\{k_{N,r}(t), \ t \in I\}$ of interger-valued, non-decreasing and right-continuous functions, where

$$k_{N,r}(t) = \max\{k: A^2_{N,k} \le tA^2_{N,r}\} \ , \quad t \in I \ , \tag{3.1}$$

and $A^2_{N,k}$ is defined by (2.14). Consider then the process $Y_{N,r} = \{Y_{N,r}(t), \ t \in I\}$ by letting

$$Y_{N,r}(t) = A^{-1}_{N,r} L_{N,k_{N,r}(t)} \ , \quad t \in I \ . \tag{3.2}$$

Thus, $Y_{N,r}$ belongs to the space $D[0,1]$, endowed with the Skorokhod J_1-topology. We propose the following two test statistics:

(i) Kolmogorov-Smirnov (KS-) type statistics. We define

$$K^*_{N,r} = \sup_{t\in I}|Y_{N,r}(t)|$$

$$= \max_{0\le k\le r}\{A^{-1}_{N,r}L_{N,k}\} \tag{3.3}$$

$$= A^{-1}_{N,r}\left\{\max_{0\le k\le r} \tilde{T}'_{N,k}\tilde{C}^{-1}_{N}\tilde{T}_{N,k})^{\frac{1}{2}}\right\}.$$

Note that, if we let

$$K^{(k)}_{N,r} = A^{-1}_{N,r}\left\{\max_{0\le q\le k}(\tilde{T}'_{N,q}\tilde{C}^{-1}_{N}\tilde{T}_{N,q})^{\frac{1}{2}}\right\},\quad 0\le k\le r, \tag{3.4}$$

then, by definition,

$$0 = K^{(0)}_{N,r} \le K^{(1)}_{N,r} \le \dots \le K^{(r)}_{N,r} = K^*_{N,r}. \tag{3.5}$$

(ii) Cramér-von Mises (CvM-) type statistics. Here, we define

$$M^*_{N,r} = \int_0^1 Y^2_{N,r}(t)dt. \tag{3.6}$$

If, we let $t^{(k)}_{N,r} = A^2_{N,k}/A^2_{N,r}$, $0\le k\le r$, and define

$$\lambda^*_k = t^{(k+1)}_{N,r} - t^{(k)}_{N,r} = A^{-2}_{N,r}[(N-k-1)/(N-1)(N-k)][a_N(k+1)-a^*_N(k+1)]^2, \tag{3.7}$$

$0\le k\le N-1$, then from (3.1), (3.2), (3.6) and (3.7), we have

$$M^*_{N,r} = \sum_{k=0}^{r-1}\lambda^*_k L^2_{Nk}/A^2_{N,r}$$

$$= A^{-2}_{N,r}\sum_{k=0}^{r-1}\lambda^*_k(\tilde{T}'_{n,k}\tilde{C}^{-1}_{N}\tilde{T}_{N,k}), \tag{3.8}$$

where the λ^*_k also depend on N,r. Here also, if we let

$$M^{(k)}_{N,r} = A^{-2}_{N,r}\sum_{s=0}^{k-1}\lambda^*_s(\tilde{T}'_{N,s}\tilde{C}^{-1}_{N}\tilde{T}_{N,s}) = \int_0^{t^{(k)}_{N,r}} Y^2_{N,r}(t)dt,\quad 0\le k\le r, \tag{3.9}$$

we obtain that

$$0 = M^{(0)}_{N,r} \le M^{(1)}_{N,r} \le \dots \le M^{(r)}_{N,r} = M^*_{N,r}. \tag{3.10}$$

The monotonicity property in (3.5) or (3.10) is then incorporated in the formulation of the following PCS tests. For a preassigned level of significance $\alpha(0 < \alpha < 1)$, let $K^*_{N,r,\alpha}$ and $M^*_{N,r,\alpha}$ be defined by

$$P\{K^*_{N,r} \geq K^*_{N,r,\alpha} | H_0\} \geq \alpha > P\{K^*_{N,r} > K^*_{N,r,\alpha} | H_0\} ; \tag{3.11}$$

$$P\{M^*_{N,r} \geq M^*_{N,r,\alpha} | H_0\} \geq \alpha > P\{M^*_{N,r} > M^*_{N,r,\alpha} | H_0\} . \tag{3.12}$$

Then, for N items under life testing, as the successive failures are observed, at each failure Z_{Nk}, we compute $K^{(k)}_{N,r}$ (or $M^{(k)}_{N,r}$), $k \geq 0$. If, for the first time, for some $k(\leq r)$, $K^{(k)}_{N,r}$ is $> K^*_{N,r,\alpha}$ (or $M^{(k)}_{N,r}$ is $> M^*_{N,r,\alpha}$), experimentation is discontinued following Z_{Nk} along with the rejection of H_0. If, no such $k(\geq r)$ exists, experimentation is curtailed following Z_{Nr} along with the acceptance of H_0. Note that, by definition, $K^*_{N,r}$ and $M^*_{N,r}$ are both functions of $\underset{\sim}{R}_N$ (and $\underset{\sim}{c}_1, \ldots, \underset{\sim}{c}_N$), and hence, under H_0 in (2.2), they are distribution-free statistics. Thus, both the PCS tests based on $\{K^{(k)}_{N,r}\}$ and $\{M^{(k)}_{N,r}\}$ are genuinely distribution-free (under H_0). Also, by (2.13), (2.19), (3.3) and (3.8), these statistics remain invariant under any non-singular transformation on the regression vectors $\{\underset{\sim}{c}_i\}$. That is, if we let $\underset{\sim}{d}_i = \underset{\sim}{c}_0 + \underset{\sim}{\Gamma}\underset{\sim}{c}_i$, $i \geq 1$, where $\underset{\sim}{c}_0$ is arbitrary and Γ is non-singular, and for (2.1), we rewrite $x - \beta_0 - \underset{\sim}{\beta}'\underset{\sim}{c}_i$ as $x - \gamma_0 - \underset{\sim}{\gamma}'\underset{\sim}{d}_i$, $1 \leq i \leq N$, where $\underset{\sim}{\gamma}' = \underset{\sim}{\beta}'\Gamma^{-1}$ and $\gamma_0 = \beta_0 - \underset{\sim}{\gamma}'\underset{\sim}{c}_0$, then replacing in (2.11), (2.13) and (2.19), the $\underset{\sim}{c}_i$ by $\underset{\sim}{d}_i$ and denoting the resulting statistics by $\tilde{L}_{N,k}$, we have $L_{N,k} = \tilde{L}_{N,k}$, $\forall\, k \geq 0$ and $\underset{\sim}{\Gamma}$ (non-singular). Since, in many cases (viz., the multipsample location problem), the $\underset{\sim}{c}_i$ and $\underset{\sim}{\beta}$ in (2.1) are not uniquely defined, this invariance is rather important and it eliminates the arbitrariness of the choice of the $\underset{\sim}{c}_i$. Thus, the proposed PCS tests are invariant, distribution-free tests. For small values of N, the exact null distribution of $K^*_{N,r}$ or $M^*_{N,r}$ can be derived by direct enumerations of the exact distribution of $\underset{\sim}{R}_N$ (or $\underset{\sim}{S}_N$) over the $N!$ equally likely permutations of $(1,\ldots,N)$ giving rise to $N^{[r]}$ equally likely realizations of $(S_{N1},\ldots,S_{Nr})$ from $(1,\ldots,N)$. The task becomes prohibitively laborious as r (and hence, N) increases; for this reason, we take recourse to the asymptotic distribution in the next section.

4. ASYMPTOTIC NULL DISTRIBUTION THEORY

To study the asymptotic distributions of $K^*_{N,r}$ and $M^*_{N,r}$ (both of which are functionals of the process $Y_{N,r}$, defined in (3.1)-(3.2)), first, we consider the weak convergence of $\{Y_{N,r}\}$ to appropriate functionals of Brownian motions. Let $W_j = \{W_j(t),\ t \in I\}$, $j = 1,\ldots,p$ be p independent copies of standard Brownian motions on I, and define $Y = \{Y(t),\ t \in I\}$ by letting

$$Y(t) = \left[\sum_{j=1}^{p} W_j^2(t)\right]^{\frac{1}{2}}, \quad t \in I . \tag{4.1}$$

Then, Y belongs to the space $C[0,1]$ with probability 1. At this stage, we introduce the following (Noether-type) condition:

$$\lim_{N\to\infty} \sup \left\{ \max_{1\le i\le N} (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_N)' \underset{\sim}{C}_N^{-1} (\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_N) \right\} = 0 . \tag{4.2}$$

Then, we have the following

Theorem 4.1. *Under* H_0 *in* (2.2), (2.21), (4.2) *and the assumptions made on the scores in Section 2, as* $N \to \infty$,

$$Y_{N,r} \xrightarrow{\mathcal{D}} Y \ , \quad \textit{in the } J_1\textit{-topology on } D[0,1] \ . \tag{4.3}$$

Proof. We have noticed in the last section that the $L_{N,k}$ (and hence, $Y_{N,r}$) remain invariant under any non-singular transformation on the $\underset{\sim}{c}_i$. Thus, under (2.16), there exists a non-singular matrix $\underset{\sim}{E}_N$ such that

$$\underset{\sim}{E}_N' \underset{\sim}{C}_N \underset{\sim}{E}_N = \underset{\sim}{I}_p = \mathrm{Diag}(1,\dots,1) \ . \tag{4.4}$$

Let then

$$\tilde{\underset{\sim}{c}}_i = \underset{\sim}{E}_N(\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_N), \ 1 \le i \le N, \quad \text{so that} \quad \sum_{i=1}^{N} \tilde{\underset{\sim}{c}}_i = 0, \ \sum_{i=1}^{N} \tilde{\underset{\sim}{c}}_i \tilde{\underset{\sim}{c}}_i' = \underset{\sim}{I}_p \ . \tag{4.5}$$

Further, in (2.13), replacing $\underset{\sim}{c}_i - \bar{\underset{\sim}{c}}_N$ by $\tilde{\underset{\sim}{c}}_i$, $1 \le i \le N$, the resulting vector is denoted by $\tilde{T}_{N,k}$ for $k = 0,1,\dots,N$. Then, by (2.19), (4.4) and (4.5), we have

$$L_{N,k} = (\tilde{\underset{\sim}{T}}'_{N,k} \tilde{\underset{\sim}{T}}_{N,k})^{\frac{1}{2}} \ , \quad \text{for} \quad k = 0,1,\dots,N \ . \tag{4.6}$$

Defining $\{k_{N,r}(t),\ t \in I\}$ by (3.1), we introduce a p-variate stochastic process $\tilde{\underset{\sim}{W}}_{N,r} = \{\tilde{\underset{\sim}{W}}_{N,r}(t),\ t \in I\} = \{(\tilde{W}^{(1)}_{N,r}(t),\dots,\tilde{W}^{(p)}_{N,r}(t))',\ t \in I\}$ by letting

$$\tilde{\underset{\sim}{W}}_{N,r}(t) = A^{-1}_{N,r} \tilde{\underset{\sim}{T}}_{N,k_{N,r}(t)}, \quad t \in I \ . \tag{4.7}$$

Then, by (3.2), (4.6) and (4.7), we obtain that

$$Y_{N,r}(t) = ([\tilde{\underset{\sim}{W}}_{N,r}(t)]'[\tilde{\underset{\sim}{W}}_{N,r}(t)])^{\frac{1}{2}}, \quad t \in I \ . \tag{4.8}$$

Hence, if we let $\underset{\sim}{W} = \{(W_1(t),\dots,W_p(t))',\ t \in I\}$ where the W_j are defined prior to (4.1), then to prove (4.3), it suffices to show that

$$\tilde{\underset{\sim}{W}}_{N,r} \xrightarrow{\mathcal{D}} \underset{\sim}{W}\ , \quad \text{in the } J_1\text{-topology on } D[0,1]\ , \tag{4.9}$$

and for this, we need to show that (a) the finite-dimensional distributions (f.d.d.) of $\{\tilde{\underset{\sim}{W}}_{N,r}\}$ converge to those of $\underset{\sim}{W}$ and (b) $\{\tilde{\underset{\sim}{W}}_{N,r}\}$ is *tight*.

Note that under the assumed regularity conditions on the scores, by Lemma 4.3 of Chatterjee and Sen (1973) and our (3.7),

$$\max_{0 \le k \le r} \lambda_k^* \to 0 \quad \text{as} \quad N \to \infty\ . \tag{4.10}$$

As a consequence, $\lim_{N\to\infty} t_{N,r}^{(k_{N,r}(t))} = t$ for every t belonging to $[0,1]$. Hence, if for $q(\ge 1)$ and $0 \le t_1 < \dots < t_q \le 1$, $\underset{\sim}{t} = (t_1,\dots,t_q)'$, then by (4.4), (4.5), (2.17) and (2.18), we have on letting $\tilde{\underset{\sim}{W}}_{N,r}(t) = [\tilde{\underset{\sim}{W}}'_{N,r}(t_1),\dots,\tilde{\underset{\sim}{W}}'_{N,r}(t_q)]$ that

$$E\tilde{\underset{\sim}{W}}_{N,r}(\underset{\sim}{t}) = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{V}[\tilde{\underset{\sim}{W}}_{N,r}(\underset{\sim}{t})] \to \underset{\sim}{I}_p \otimes ((t_j \wedge t_\ell))\ , \tag{4.11}$$

where $\underset{\sim}{V}[\ ]$ stands for the dispersion matrix (of order $pq \times pq$) and $\otimes$ for the Kronecker product. Defining $\underset{\sim}{W}(\underset{\sim}{t}) = [\underset{\sim}{W}'(t_1),\dots,\underset{\sim}{W}'(t_q)]$, it follows by routine steps that

$$E\underset{\sim}{W}(\underset{\sim}{t}) = \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{V}[\underset{\sim}{W}(\underset{\sim}{t})] = \underset{\sim}{I}_p \otimes ((t_j \wedge t_\ell))\ . \tag{4.12}$$

Thus, to prove the convergence of f.d.d.'s of $\{\tilde{\underset{\sim}{W}}_{N,r}\}$ to those of $\underset{\sim}{W}$, we need to show only that for any given $q(\ge 1)$ and $\underset{\sim}{t} = (t_1,\dots,t_q)'$, $\tilde{\underset{\sim}{W}}_{N,r}(\underset{\sim}{t})$ is asymptotically normal. Writing

$$k_{N,r}(\underset{\sim}{t}) = (k_{N,r}(t_1),\ldots,k_{N,r}(t_q))' = (k_1,\ldots,k_q)' , \tag{4.13}$$

and noting (4.7) and (4.10), it suffices to show that $(\underset{\sim}{\tilde{T}}_{N,k_1},\ldots,\underset{\sim}{\tilde{T}}_{N,k_q})$ has asymptotically a multinormal distribution. We consider the rolled out vector

$$\underset{\sim}{U}'_{N,q} = (\tilde{T}^{(1)}_{N,k_1},\ldots,\tilde{T}^{(p)}_{N,k_1},\ldots,\tilde{T}^{(1)}_{N,k_q},\ldots,\tilde{T}^{(p)}_{N,k_q}) \tag{4.14}$$

and for a non-null $\underset{\sim}{g} = (g_{11},\ldots,g_{p1},\ldots,g_{1q},\ldots,g_{pq})$, consider the linear compound

$$U^*_N(g) = \underset{\sim}{g}'\underset{\sim}{U}_{N,q} = \sum_{i=1}^N \sum_{m=1}^p \tilde{c}_{mS_{Ni}} d_N(i) , \tag{4.15}$$

where, by virtue of (2.13), (4.5) and (4.15),

$$d_N(i) = \begin{cases} \left(\sum_{m=1}^p \sum_{j=1}^q g_{mj}\right) a_N(i), \quad 1 \le i \le k_1 ; \\ \left(\sum_{m=1}^p \sum_{j=1}^q g_{mj}\right) a_N(i) + \sum_{m=1}^p \sum_{j=1}^{s-1} g_{mj} \left[\frac{1}{N-k_j} \sum_{v=k_j+1}^{N} a_N(v)\right] \\ \qquad\qquad \text{for} \quad k_{s-1} < i \le k_s, \; s = 2,\ldots,q ; \\ \sum_{m=1}^p \sum_{j=1}^q g_{mj} \left[\frac{1}{N-k_j} \sum_{v=k_j+1}^{N} a_N(v)\right], \quad k_q \le i \le N . \end{cases} \tag{4.16}$$

Thus, if we let

$$f_{S_{Ni}} = \sum_{m=1}^p \tilde{c}_{mS_{Ni}} \quad \text{for} \quad i = 1,\ldots,N , \tag{4.17}$$

we have from (4.15)-(4.17) that

$$U^*_N(g) = \sum_{i=1}^N f_{S_{Ni}} d_N(i) , \tag{4.18}$$

where under H_0 in (2.2), $\underset{\sim}{S}_N = (S_{N1},\ldots,S_{NN})'$ assumes all possible permutations of $(1,\ldots,N)$ with equal probability $(N!)^{-1}$. Thus, $U_N^*(g)$ is a simple linear (anti-) rank statistic. Note that by assumptions (2.3)-(2.7) and by Lemma 2.2 and Theorem 3.2 of Hajék (1961), the $a_N(i)$ satisfy the condition Q of Hajék (1961), i.e.,

$$\lim_{N\to\infty} N^{-1}\ell_N = 0 \Longrightarrow \lim_{N\to\infty}\left\{\max_{1\le i_1<\ldots<i_{\ell_N}\le N} N^{-1}\sum_{\alpha=1}^{\ell_N}[a_N(i_\alpha)-\bar{a}_N]^2\right\} = 0 . \tag{4.19}$$

Since, the g_{mj} are real constants, by (4.16) and (4.19), it follows by some standard steps that

$$\lim_{N\to\infty} N^{-1}\ell_N = 0 \Longrightarrow \lim_{N\to\infty}\left\{\max_{1\le i_1<\ldots<i_{\ell_N}\le N} N^{-1}\sum_{\alpha=1}^{\ell_N}[d_N(i_\alpha)-\bar{d}_N]^2\right\} = 0 , \tag{4.20}$$

where $\bar{d}_N = N^{-1}\sum_{\alpha=1}^{N} d_N(\alpha) = 0$. Also, by (2.3)-(2.7) and (4.16), it follows that for every $\underset{\sim}{g} \ne \underset{\sim}{0}$, there exists a finite and positive g^*, such that

$$N^{-1}\sum_{i=1}^{N}[d_N(i)-\bar{d}_N]^2 \to g^* \quad \text{as} \quad N\to\infty . \tag{4.21}$$

Thus, the $d_N(i)$ satisfy the condition Q of Hajék (1961). Further, by (4.17), $\sum_{i=1}^{N} f_i^2 = \sum_{i=1}^{N}\sum_{m=1}^{p}(\tilde{c}_{mi})^2 = \sum_{i=1}^{N}\tilde{\underset{\sim}{c}}_i'\tilde{\underset{\sim}{c}}_i = p$ (see (4.5)) and by (4.2), (4.4), (4.5) and (4.17),

$$\max_{1\le i\le N} f_i^2 \to 0 \quad \text{as} \quad N\to\infty \tag{4.22}$$

so that the f_i satisfy the Noether condition. Hence, the asymptotic normality of (4.18) follows directly by an appeal to the Hajék (1961) theorem, and the convergence of f.d.d.'s of $\{\tilde{\underset{\sim}{W}}_{N,r}\}$ to $\underset{\sim}{W}$ holds.

Chatterjee and Sen (1973) have established the tightness of $\tilde{\underset{\sim}{W}}_{N,r}$ for the special case of $p=1$. As such, using their result coordinate wise, we obtain that for every $\varepsilon > 0$ and $\eta > 0$, there exists a δ: $0 < \delta < 1$ and an integer N_0, such that for $N \geq N_0$ and each $j(=1,\ldots,p)$,

$$P\{\sup[|\tilde{W}_{N,r}^{(j)}(t)-\tilde{W}_{N,r}^{(j)}(s)|: 0 \leq s \leq t \leq s+\delta \leq 1] > \varepsilon p^{-\frac{1}{2}}\} < p^{-1}\eta . \tag{4.23}$$

On the other hand, $||\tilde{\underset{\sim}{W}}_{N,r}(t)-\tilde{\underset{\sim}{W}}_{N,r}(s)||^2 = \sum_{j=1}^{p}[\tilde{W}_{N,r}^{(j)}(t)-\tilde{W}_{N,r}^{(j)}(s)]^2$, so that by (4.23),

$$P\{\sup[||\tilde{\underset{\sim}{W}}_{N,r}(t)-\tilde{\underset{\sim}{W}}_{N,r}(s)||: 0 \leq s \leq t \leq s+\delta \leq 1] > \varepsilon\} < \eta, \quad \forall\ N \leq N_0 , \tag{4.24}$$

while $\tilde{\underset{\sim}{W}}_{N,r}(0) = \underset{\sim}{0}$ with probability 1 (by definition). Hence, the proof of the tightness of $\{\tilde{\underset{\sim}{W}}_{N,r}\}$ is complete. Q.E.D.

Let now $\mathcal{B}_{N,k}$ be the signma-field generated by $\underset{\sim}{S}_N^{(k)} = (S_{N1},\ldots,S_{Nk})$, $0 \leq k \leq N$, $\mathcal{B}_{N,0}$ being the trivial sigma-field. Then, for every $N(\geq 1)$, $\mathcal{B}_{N,k}$ is non-decreasing in k. Further, we define N_0 as in (2.10).

<u>Lemma 4.2</u>. *Under* H_0 *in* (2.2), $\{\underset{\sim}{T}_{N,k}, \mathcal{B}_{N,k}; 0 \leq k \leq N\}$ *is a martingale for every* $N(\geq 1)$, $\{L_{N,k}, \mathcal{B}_{N,k}; 0 \leq k \leq N\}$ *is a submartingale.*

<u>Proof</u>. Under H_0 in (2.2), by Lemma 4.1 of Chatterjee and Sen (1973), it follows directly that for each j $(=1,\ldots,p)$,

$$E(T_{N,k+1}^{(j)}|\mathcal{B}_{N,k}) = T_{N,k}^{(j)} \quad \text{a.e. for every } k: 0 \leq k \leq N-1 , \tag{4.25}$$

and hence, $E(\underset{\sim}{T}_{N,k+1}|\mathcal{B}_{N,k}) = \underset{\sim}{T}_{N,k}$ a.e., $\forall\ 0 \leq k \leq N-1$. The same is true for $\tilde{\underset{\sim}{T}}_{N,k}$. Further by (2.13), (2.15), (2.16) and (2.19), $L_{N,k}$ (being the Euclidean norm) is convex in $\tilde{\underset{\sim}{T}}_{N,k}$, and hence, the martingale property of $\{\tilde{\underset{\sim}{T}}_{N,k}\}$ along with the conditional form of the Jensen inequality yields the submartingale property of $\{L_{N,k}\}$. Q.E.D.

For every t: $0 < t < 1$, let us define now

$$\nu(t) = \int_0^t \phi^2(u)\,du + (1-t)^{-1}\left(\int_t^1 \phi(u)\,du\right)^2 , \qquad (4.26)$$

so that $\nu(t)$ is $\nearrow$ in $t \in (0,1)$, $\nu(0) = \lim_{t \downarrow 0} \nu(t) = 0$ and $\nu(1) = \lim_{t \uparrow 1} \nu(t) = A_\phi^2 = 1$.

Lemma 4.3. *Under* (2.3)-(2.7), $k/N \to t$: $t \in [0,1]$ *insures that*

$$A_{N,k}^2 = (N-1)^{-1}\sum_{i=1}^N b_{N,k}^2(i) \to \nu(t) \ , \quad \text{as} \quad N \to \infty , \qquad (4.27)$$

where the $b_{N,k}(i)$ *are defined in* (2.12).

The proof follows along the lines of the proof of Lemma 4.2 and Theorem 4.2 of Chatterjee and Sen (1973), and hence, is omitted.

Let us now introduce the following:

$$\omega_p^* = \sup_{t \in I} |Y(t)| = \sup_{t \in I}\left[\sum_{j=1}^p W_j^2(t)\right]^{\frac{1}{2}} , \qquad (4.28)$$

$$\omega_p^0 = \int_0^1 Y^2(t)\,dt = \sum_{j=1}^p \int_0^1 W_j^2(t)\,dt = \sum_{j=1}^p \omega_j \ , \qquad (4.29)$$

where the $\omega_j \left(= \int_0^1 W_j^2(t)\,dt\right)$ are i.i.d. nonnegative r.v. From Theorem 4.1 we conclude that under the hypothesis of Theorem 4.1,

$$M_{N,r}^* \xrightarrow{\mathcal{D}} \omega_p^0 \quad \text{and} \quad K_{N,r}^* \xrightarrow{\mathcal{D}} \omega_p^* \quad \text{as} \quad N \to \infty . \qquad (4.30)$$

The characteristic function (c.f.) $g(\theta)$ of ω_1 (or any ω_j, $j \geq 1$) is given by [viz., Dugue (1969)]

$$g(\theta) = \left(\mathrm{Cos}\sqrt{2i\theta}\right)^{-\frac{1}{2}} = \prod_{k=1}^{\infty}\{1 - 2i\theta u_k\}^{-\frac{1}{2}}; \ u_k = \frac{4}{\pi^2(2k-1)^2}, \ k \geq 1 . \qquad (4.31)$$

Therefore, the c.f. $g_p^0(\theta)$ of ω_p^0 is given by

$$g_p^0(\theta) = [g(\theta)]^p = \prod_{k=1}^{\infty}\{1-2i\theta u_k\}^{-p/2} . \tag{4.32}$$

Note that if $\{U_j, j \geq 1\}$ be a squence of i.i.d.r.v. where U_j has the central chi-square distribution with p degrees of freedom, then for every c,

$$E\{\exp[itcU_j]\} = [1-2itc]^{-p/2} , \tag{4.33}$$

so that from (4.32) and (4.33), we have

$$\omega_p^0 \overset{\mathcal{D}}{=} \sum_{k=1}^{\infty}\{4/\pi^2(2k-1)^2\}U_k , \tag{4.34}$$

where $\overset{\mathcal{D}}{=}$ stands for the equality of distributions. From (4.30) and (4.34), we conclude that under H_0, as $N \to \infty$,

$$M_{N,r}^* \overset{\mathcal{D}}{\to} \sum_{k=1}^{\infty}\{4/\pi^2(2k-1)^2\}U_k = U_0^0 , \text{ say} . \tag{4.35}$$

Since, we do not know the distribution of U_0^0 in any closed form, we have obtained (by simulation studies) the empirical percentile points: these will be reported in Section 7.

For $p = 1$, the distribution of $\omega_1^* = \sup_{t\in I}|W_1(t)|$ is well-known and is given by [viz., Billingsley (p.79; 1968)]

$$P\{\omega_1^* \leq x\} = \sum_{k=-\infty}^{\infty}(-1)^k[\Phi((2k+1)x) - \Phi((2k-1)x)] , \quad x \geq 0 , \tag{4.36}$$

where Φ is the standard normal d.f. Parallel expressions for $p > 1$ are not known and remain as challenging problems for probabilists. In Section

7, we have also derived, through simulation studies, some empirical values for the percentile points of the d.f. of ω_p^* for $p \le 4$.

We may however, note that, by definition, $\{Y(t),\ t \in I\}$ is a submartingale, so that for every $x > 0$,

$$P\left\{\sup_{t\in I} Y(t) > x\right\} = P\left\{\sup_{t\in I} e^{\theta Y^2(t)} > e^{\theta x^2}\right\} \quad (\theta > 0)$$

$$\le \inf_{\theta>0}\left\{e^{-\theta x^2} E(e^{\theta Y^2(1)})\right\}$$

$$= \inf_{\theta>0}\left\{e^{-\theta x^2}(1-2\theta)^{-p/2}\right\} \tag{4.37}$$

as $Y^2(1) = \sum_{j=1}^{p} W_j^2(1)$ has the central chi-square d.f. with p degrees of freedom; note that the inequality in (4.37) is based on the Kolmogorov inequality for submartingales. For $x^2 > p$, the right hand side reduces to

$$(e/p)^{p/2} x^p e^{-\frac{1}{2}x^2} \left(\simeq \sqrt{2\pi}\{\sqrt{p/2}\ 2^{p/2}\Gamma(p/2)\}^{-1} x^p e^{-\frac{1}{2}x^2}\right) . \tag{4.38}$$

On the other hand, as $\omega_p^* \ge \omega_1^*$, we have by (4.36), for every $x > 0$,

$$P\{\omega_p^* > x\} \ge \{\omega_1^* > x\} = 1 - P\{\omega_1^* \le x\}$$

$$= 2[1-\Phi(x)] + 2\sum_{k=1}^{\infty}(-1)^{k-1}[\Phi((2k+1)x)-\Phi((2k-1)x)] \tag{4.39}$$

$$(\sim 4[1-\Phi(x)] \text{ when } x \text{ is not small}) .$$

From (4.37)-(4.39), we obtain that

$$\lim_{x\to+\infty}[-2x^{-2}\log P\{\omega_p^* > x\}] = 1 , \quad \text{for every } p \ge 1 . \tag{4.40}$$

We shall find this result quite useful in Section 6. In passing, we may remark that in (4.37)-(4.38), instead of using the Kolmogorov inequality, we could have used the Doob upcrossing inequality for submartingales and obtain the upper bound

$$\inf_{\theta>0}\left\{2e^{-\theta x^2}E[e^{\theta Y^2(1)}I(e^{\theta Y^2(1)}>\tfrac{1}{2}e^{\theta x^2})]\right\}$$

$$=\inf_{\theta>0}\left\{2e^{-\theta x^2}\frac{1}{2^{p/2}\Gamma(p/2)}\int_{u>-\frac{1}{\theta}\log 2+x^2}u^{\frac{p}{2}-1}e^{-\frac{1}{2}(1-2\theta)u}du\right\} \tag{4.41}$$

$$=\inf_{\theta>0}\left\{e^{-\theta x^2}2(1-2\theta)^{-p/2}2^{-p/2}(\Gamma(p/2))^{-1}\int_{v>(1-2\theta)[x^2-\frac{1}{\theta}\log 2]}v^{p/2-1}e^{-\frac{1}{2}v}dv\right\}.$$

For specific values of p (viz., $p=2$), the right hand side of (4.41) can be worked out explicitly and the same is somewhat sharper than (4.38).

5. ASYMPTOTIC NON-NULL DISTRIBUTION THEORY

With the intention of studying the (asymptotic) power properties of the proposed PCS tests, we proceed now to consider the asymptotic non-null distribution of $K^*_{N,r}$ and $M^*_{N,r}$. For fixed alternative hypothesis, these distributions do not exist and we are left with the task of studying the rates of convergence of the powers to 1 — as will be done in the next section. On the other hand, as is the usual fashion, we may consider a sequence of local aternative hypotheses, chosen so carefully that under such a case, the asymptotic non-null distributions are properly defined and the powers are bounded away from 1.

Towards the end of Section 3, we have observed that $K^*_{N,r}$ and $M^*_{N,r}$ are both invariant under non-singular transformations on the regression vectors — hence, we may, without any loss of generality, use a canonical reduction. We assume that there is a triangular array $\{X_{Ni},\ 1 \le i \le N,\ N \ge 1\}$ of rowwise independent r.v., where for each N,

$$P\{X_{Ni} \le x\} = F(x - \beta_0 - \underset{\sim}{\beta}' \underset{\sim}{c}^*_{Ni})\ ,\quad 1 \le i \le N\ ,\ -\infty < x < \infty\ , \tag{5.1}$$

where $\underset{\sim}{c}^*_{Ni} = (c^*_{N1i}, \ldots, c^*_{Npi})'$, $i = 1, \ldots, N$, and

$$\textstyle\sum_{i=1}^{N} \underset{\sim}{c}^*_{Ni} = \underset{\sim}{0} \quad \text{and} \quad \sum_{i=1}^{N} \underset{\sim}{c}^*_{Ni} \underset{\sim}{c}^{*\prime}_{Ni} = \underset{\sim}{I}_p\ , \tag{5.2}$$

and in this case, (4.2) reduces to $\max_{1 \le j \le p} \left\{ \max_{1 \le i \le N} c^{*2}_{Nji} \right\} \to 0$. As before, H_0: $\underset{\sim}{\beta} = \underset{\sim}{0}$ and we frame a sequence $\{H_N\}$ of alternative hypotheses by letting

$$H_N:\ (5.1)\text{-}(5.2) \ \text{ hold with } \ \underset{\sim}{\beta} \ne \underset{\sim}{0}\ . \tag{5.3}$$

Regarding F in (5.1), we assume that it has an absolutely continuous density function f with a finite Fisher information

$$I(f) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 dF(x)\ ,\quad \text{where} \quad f'(x) = \frac{d}{dx} f(x)\ . \tag{5.4}$$

In addition, we define

$$\phi^0(u) = -f'(F^{-1}(u))/f(F^{-1}(x))\ ,\quad 0 < u < 1\ , \tag{5.5}$$

so that $\int_0^1 \phi^0(u)du = 0$ and $\int_0^1 [\phi^0(u)]^2 du = I(f) < \infty$. Also, for $s \in (0,1)$, we let

$$\phi_s(u) = \begin{cases} \phi(u)\ , & 0 < u \le s\ , \\ \phi^*_s = (1-s)^{-1} \int_s^1 \phi(t)dt\ , & s < u < 1\ , \end{cases} \tag{5.6}$$

and define $\phi_s^0(u)$, $0 < u < 1$ in the same way. Let then for $s \in (0.1)$,

$$\rho(s) = \left(\int_0^1 \phi_s(u)\phi^0(u)\,du\right)/I^{\frac{1}{2}}(f) = \left(\int_0^1 \phi_s^0(u)\phi_s^0(u)\,du\right)/I^{\frac{1}{2}}(f) \ , \qquad (5.6)$$

and finally, we define $\nu(s)$, $0 \le s \le 1$ as in (4.26). Then, the follow- theorem provides the bases for subsequent results of this section.

Theorem 5.1. *Under* (5.1)-(5.4) *and the conditions on the score function* ϕ *assumed in Section* 2, $\tilde{\underset{\sim}{W}}_{N,r}$, *defined by* (4.7) [*but for the triangular array of r.v.'s in* (5.1)], *converges in law in the* J_1*-topology on* $D[0,1]$ *to a p-variate Gaussian function* $\underset{\sim}{W} + \underset{\sim}{\mu}$, *where* $\underset{\sim}{W}$ *is defined before* (4.9) *and* $\underset{\sim}{\mu} = \{\underset{\sim}{\mu}(t),\ t \in I\}$ *is given by*

$$\underset{\sim}{\mu}(t) = \underset{\sim}{\beta}\rho(\nu^{-1}(t\nu(\delta)))[I(f)/\nu(\delta)]^{\frac{1}{2}} \ , \quad 0 \le t \le 1 \ . \qquad (5.8)$$

Proof. Let P_N^0 and P_N be respectively the joint d.f. of $(X_{N1},\ldots,X_{NN})$ under H_0 and H_N and let P_{Nk}^0 and P_{Nk} be the same for $(Z_{N1},\ldots,Z_{Nk})$, for $k \le N$. Then, by the results of Chapter VI of Hájek and Šidak (1967), we conclude that under (5.1)-(5.4), $\{P_N\}$ is contiguous to $\{P_N^0\}$, and this insures that $\{P_{Nk},\ k \le N\}$ is also contiguous to $\{P_{Nk}^0,\ k \le N\}$. Hence, we may proceed along the lines of the proof of Theorem 2 of Sen (1976a) and show that the tightness of $\tilde{\underset{\sim}{W}}_{N,r}$, under H_0, insures the same under the contiguous alternatives $\{H_N\}$. The convergence of f.d.d.'s of $\tilde{\underset{\sim}{W}}_{N,r} - \underset{\sim}{\mu}$ to those of $\underset{\sim}{W}$ also follows by an appeal to contiguity (when $\{H_N\}$ holds) and the earlier part of the proof of Theorem 4.1 along the same line as in Theorem 2 of Sen (1976a). Q.E.D.

Recalling that $\underset{\sim}{W}(t) = (W_1(t),\ldots,W_p(t))'$, $t \in I$ and $\underset{\sim}{\mu}(t) = (\mu_1(t),\ldots,\mu_p(t))'$, $t \in I$, we obtain from (3.3), (3.6), (4.7), (4.8) and Theorem 5.1 that under the hypothesis of Theorem 5.1, (under $\{H_N\}$),

$$K^*_{N,r} \xrightarrow{\mathcal{D}} \sup_{t \in I}\left\{\left[\sum_{j=1}^{p}(W_j(t)+\mu_j(t))^2\right]^{\frac{1}{2}}\right\}, \tag{5.9}$$

$$M^*_{N,r} \xrightarrow{\mathcal{D}} \sum_{j=1}^{p}\int_0^1 \{W_j(t)+\mu_j(t)\}^2 dt \ . \tag{5.10}$$

Note that the $\mu_j(t)$, defined by (5.8), *are not, in general, linear functions of* t, and as in Section 4, the exact distributions for the right hand sides of (5.9) and (5.10) are difficult to obtain. As a result, it is difficult to draw more indepth conclusions about the relative performance of these PCS tests for contiguous alternatives. For this reason, in the next section, we take recourse to the Bahadur efficiency, where under (fixed but) close alternatives, we have some meaningful comparisons of the different test statistics and score functions.

6. BAHADUR A.R.E. OF PCS TESTS

First, parallel to (4.40), we derive a limiting result for the tail probability of the Cramér-von Mises type statistics. Note that by (4.35),

$$\frac{4}{\pi^2} U_1 < U_0^0 = \frac{4}{\pi^2} U_1 + \sum_{k=2}^{\infty}\{4/\pi^2(2k-1)^2\}U_k \ , \tag{6.1}$$

where each U_j, $j \ge 1$, has the central chi-square distribution with p degrees of freedom, so that for every $\lambda > 0$,

$$P\{U_j > \lambda^2\} = (2^{p/2}\Gamma(p/2))^{-1}\int_{\lambda^2}^{\infty} e^{-\frac{1}{2}x^2}(x^2)^{p/2-1}dx^2 . \tag{6.2}$$

Some standard analysis on (6.2) leads us to

$$\lim_{\lambda\to\infty}\left[-\frac{2}{\lambda^2}\log P\{U_j > \lambda^2\}\right] = 1 , \tag{6.3}$$

so that from (6.1) and (6.3), we obtain that

$$\limsup_{\lambda\to\infty}\left[-\frac{2}{\lambda^2}\log P\{U_0^0 > \lambda^2\}\right]$$

$$\leq \limsup_{\lambda\to\infty}\left[-\frac{2}{\lambda^2}\log P\{U_1 > \lambda^2\pi^2/4\}\right] = \pi^2/4 . \tag{6.4}$$

On the other hand, by the independence of the U_j,

$$\begin{aligned}
&P\left\{\sum_{i=1}^{\infty}\{4/\pi^2(2k-1)^2\}U_k > \lambda^2\right\} \\
&= P\left\{\exp[\theta\sum_{k=1}^{\infty}\{4/\pi^2(2k-1)^2\}U_k] > \exp[\theta\lambda^2]\right\} \qquad (\theta > 0) \\
&\leq \inf_{\theta>0}\left\{e^{-\theta\lambda^2}E[\exp\theta\sum_{k=1}^{\infty}\{4/\pi^2(2k-1)^2\}U_k]\right\} \\
&= \inf_{\theta>0}\left\{e^{-\theta\lambda^2}\prod_{k=1}^{\infty}E\left(e^{\theta U_k 4/\pi^2(2k-1)^2}\right)\right\} \\
&= \inf_{\theta>0}\left\{e^{-\theta\lambda^2}\prod_{k=1}^{\infty}(1-8\theta/\pi^2(2k-1)^2)^{-p/2}\right\} \\
&\leq e^{-(\pi^2/8-\varepsilon)\lambda^2}\prod_{k=1}^{\infty}(1-(\pi^2-8\varepsilon)/\pi^2(2k-1)^2)^{-p/2} \qquad (\varepsilon > 0) \\
&= \left\{\left(1-\left(1-\frac{8\varepsilon}{\pi^2}\right)\right)^{-p/2}\prod_{k=2}^{\infty}\left(1-\frac{\pi^2-8\varepsilon}{\pi^2(2k-1)^2}\right)^{-\frac{p}{2}}\right\}e^{-\pi^2\lambda^2/8+\varepsilon\lambda^2} \\
&\leq [c(\varepsilon)]\exp\{-\pi^2\lambda^2/8+\varepsilon\lambda^2\} ,
\end{aligned} \tag{6.5}$$

where $c(\varepsilon)$ $(<\infty)$ depends on $\varepsilon(0<\varepsilon<\pi^2/8)$. Thus, choosing $\varepsilon(>0)$ arbitrarily small, we obtain from (6.5) that

$$\liminf_{\lambda\to\infty}\left[-\frac{2}{\lambda^2}\log P\{U_0^0>\lambda^2\}\right]\geq\frac{\pi^2}{4}-2\varepsilon\ ,\quad\forall\ \varepsilon(0<\varepsilon<\pi^2/8)\ , \tag{6.6}$$

so that from (6.4) and (6.6), we conclude that

$$\lim_{\lambda\to\infty}\left[-\frac{2}{\lambda^2}\log P\{U_0^0>\lambda^2\}\right]=\pi^2/4\ . \tag{6.7}$$

In some specific cases, the stochastic convergence of the $L_{N,k}$, $k\leq N$ can be studied for general alternative [viz., Chatterjee and Sen (1973) for the two sample problem]. But, in general, it demands extra regularity conditions. Let us assume that $X_1,\ldots,X_N$ are independent with d.f.'s $F_1,\ldots,F_N$ and defining $\tilde{\underset{\sim}{c}}_i$ as in (4.5), we let

$$\overline{F}_{(N)}(x)=N^{-1}\textstyle\sum_{i=1}^N F_i(x),\ \overline{\underset{\sim}{D}}_{(N)}(x)=N^{-\frac{1}{2}}\sum_{i=1}^N\tilde{\underset{\sim}{c}}_iF_i(x),\ -\infty<x<\infty\ , \tag{6.8}$$

and defining $\underset{\sim}{C}_N$ be (2.15), we assume that

$$\text{(i)}\quad \lim_{N\to\infty}N^{-1}\underset{\sim}{C}_N=\underset{\sim}{C}\ \text{and}\ \lim_{N\to\infty}\bar{\underset{\sim}{c}}_N=\bar{\underset{\sim}{c}}\ \text{ both exist ,} \tag{6.9}$$

$$\text{(ii)}\quad \lim_{N\to\infty}\overline{F}_{(N)}(x)=\overline{F}(x)\ \text{ exists for all } x\ \text{(a.e.) ,} \tag{6.10}$$

$$\text{(iii)}\quad \lim_{N\to\infty}\overline{\underset{\sim}{D}}_{(N)}(x)=\overline{\underset{\sim}{D}}(x)\ \text{ exists for all } x\ \text{(a.e.) .} \tag{6.11}$$

Let us also define $\nu(t)$ as in (4.26) and let

$$\underset{\sim}{\tau}(x)=\int_{-\infty}^{x}\phi(\overline{F}(y))d\overline{\underset{\sim}{D}}(y)-\{1-\overline{F}(x)\}^{-1}\overline{\underset{\sim}{D}}(x)\left(\int_x^\infty\phi(\overline{F}(x))d\overline{F}(x)\right)\ , \tag{6.12}$$

$$\xi(x)=(\underset{\sim}{\tau}(x))'(\underset{\sim}{\tau}(x))\ ,\quad -\infty<x<\infty\ , \tag{6.13}$$

$$\xi^*(x) = \sup_{-\infty<y\le x}[\xi(y)]^{\frac{1}{2}}, \ -\infty < x < \infty, \tag{6.14}$$

$$\xi^0(x) = \left(\int_{-\infty}^{x}\xi(y)d\nu(\bar{F}(y))\right)/\nu(\bar{F}(x)), \ -\infty < \ < \infty. \tag{6.15}$$

Then, along the lines of Section 5 of Chatterjee and Sen (1973, p. 41), it can be shown by some standard steps that under (2.3)-(2.6), (2.21) and (6.9)-(6.12), as $N \to \infty$,

$$K^*_{N,r} \to \xi^*(\bar{F}^{-1}(\delta))/\nu^{\frac{1}{2}}(\delta) \quad \text{a.s.}, \quad M^*_{N,r} \to \xi^0(\bar{F}^{-1}(\delta)) \quad \text{a.s.}, \tag{6.16}$$

$$L^2_{N,r}/A^2_{N,r} \to \xi(\bar{F}^{-1}(\delta))/\nu(\delta) \quad \text{a.s.} \tag{6.17}$$

In particular, for the model (2.1), if the $\underset{\sim}{c}_i$ have all bounded elements and $\underset{\sim}{\beta}$ is close to $\underset{\sim}{0}$, then (6.16)-(6.17) simplify to

$$K^*_{N,r} \xrightarrow{a.s.} (\underset{\sim}{\beta}'\underset{\sim}{C}^{-1}\underset{\sim}{\beta})^{\frac{1}{2}}\left\{\sup_{0\le t\le\delta}\left[\int_0^1\phi_t(u)\phi_0(u)du\right]^2/\nu(\delta)\right\}^{\frac{1}{2}} + o(||\underset{\sim}{\beta}||)$$

$$= [(\underset{\sim}{\beta}'\underset{\sim}{C}^{-1}\underset{\sim}{\beta})I(f)/\nu(\delta)]^{\frac{1}{2}}\left\{\sup_{0\le t\le\delta}\rho^2(t)\right\}^{\frac{1}{2}} + o(||\underset{\sim}{\beta}||), \tag{6.18}$$

where $\rho(s)$ is defined by (5.7) and $||\underset{\sim}{\beta}|| = \underset{\sim}{\beta}'\underset{\sim}{\beta}$;

$$M^*_{N,r} \xrightarrow{a.s.} (\underset{\sim}{\beta}'\underset{\sim}{C}^{-1}\underset{\sim}{\beta})\left[\int_{\infty}^{\bar{F}^{-1}(\delta)}\rho^2(\bar{F}(y)d\nu(\bar{F}(y))\right][I(f)/\nu^2(\delta)] + o(||\underset{\sim}{\beta}||)$$

$$= [(\underset{\sim}{\beta}'\underset{\sim}{C}^{-1}\underset{\sim}{\beta})I(f)/\nu(\delta)]\left[\int_0^{\delta}\rho^2(u)d\nu(u)\right]/\nu(\delta) + o(||\underset{\sim}{\beta}||) \qquad 6.19)$$

$$L^2_{N,r}/A^2_{N,r} \xrightarrow{a.s.} (\underset{\sim}{\beta}'\underset{\sim}{C}^{-1}\underset{\sim}{\beta})\rho^2(\delta)I(f)/\nu(\delta) + o(||\underset{\sim}{\beta}||). \tag{6.20}$$

Finally, note that under H_0 in (2.2), by Theorem 4.1, $L^2_{N,r}/A^2_{N,r} \overset{\mathcal{D}}{\to} U_1$ and hence (6.3) applies.

By virtue of Theorem 4.1, (4.30), (4.40), (6.2) (for U_1) and (6.16)-(6.17), we are in a position to adapt the Bahadur efficiency results (viz., Puri and Sen (1971, pp. 122-123). The *BARE* (Bahadur ARE) of the Komogorov-Smirnov type test relative to the terminal test based on $L_{N,r}$ is given by

$$
\begin{aligned}
e_{K,T} &= [\xi^*(\bar{F}^{-1}(\delta))]^2/\xi(\bar{F}^{-1}(\delta)) \\
&= \left\{\sup_{-\infty<x\le \bar{F}^{-1}(\delta)}[\xi(x)]\right\}/\xi(\bar{F}^{-1}(\delta)) \ge 1 , \qquad (6.21)
\end{aligned}
$$

where the equality sign holds (among other cases) when $\xi(x)$ is non-decreasing in x. Similarly, the BARE of the Cramér-von Mises type test with respect to the terminal test based on $L_{N,r}$ is

$$
\begin{aligned}
e_{M,T} &= [\xi^0(\bar{F}^{-1}(\delta))/\xi(\bar{F}^{-1}(\delta))](\pi^2/4) \\
&= \left[\left\{\int_{-\infty}^{\bar{F}^{-1}(\delta)}[\xi(y)/\xi(\bar{F}^{-1}(\delta))]dv(\bar{F}(y))\right\}/v(\delta)\right][\pi^2/4] . \qquad (6.22)
\end{aligned}
$$

Unlike (6.21), (6.22) may not be greater than or equal to one in all cases. We shall make more comment on it later on. Finally, the BARE of the Kolmogorov-Smirnov type with respect to the Cramér-von Mises type is given by

$$
\begin{aligned}
e_{K,M} &= (4/\pi^2)([\xi^*(\bar{F}^{-1}(\delta))]^2/\xi^0(\bar{F}^{-1}(\delta))) \\
&= (4/\pi^2)\left\{\sup_{-\infty<x<\bar{F}^{-1}(\delta)}\xi(x)/\left[\{v(\delta)\}^{-1}\int_{-\infty}^{\bar{F}^{-1}(\delta)}\xi(y)dv(\bar{F}(y))\right]\right\} . \qquad (6.23)
\end{aligned}
$$

An obvious lower bound for (6.23) is $4/\pi^2 = 0.4053$.

Let us now confine ourselves to local alternatives for which (6.18)-(6.20) hold and in this case, the limiting BARE reduces to

$$e^*_{M,T} = \frac{\pi^2}{4}\left(\int_0^\delta \rho^2(u)d\nu(u)\right)/[\nu(\delta)\rho^2(\delta)] \ ; \tag{6.24}$$

$$e^*_{K,M} = \frac{4}{\pi^2}\left(\sup_{0\le t\le\delta}\rho^2(t))\nu(\delta)\right)/\left(\int_0^\delta \rho^2(u)d\nu(u)\right) ; \tag{6.25}$$

$$e^*_{K,T} = \left[\sup_{0\le t\le\delta}\rho^2(t)\right]/\rho^2(\delta) \ (\ge 1) \ . \tag{6.26}$$

In the context of optimality of score functions for PCS rank tests for simple regression, Chatterjee and Sen (1973) and Sen (1976b) have studied the optimality of $\phi^0(u)$, defined by (5.5). It follows from (5.4)-(5.7) that for $\phi \equiv \phi^0$ (upto a scalar constant),

$$\nu(t) = \rho(t) = \left(\int_0^1 [\phi_t^0(u)]^2 du\right)/\int_0^1 [\phi^0(u)]^2 du \ , \quad 0 \le t \le 1 \ , \tag{6.27}$$

so that we have from (6.24)-(6.27),

$$e^*_{M,T} = \pi^2/12 = 0.8225$$

$$e^*_{K,M} = 12/\pi^2 = 1.2159$$

$$e^*_{K,T} = 1 \ .$$

In this case, we are naturally inclinced towards using the Komogorov-Smirnov type tests on the ground of the limiting BARE. However, the picture can be different when $\phi \ne \phi^0$. For example, suppose one uses the exponential score $\phi(u) = -1-\log(1-u)$, $0 < u < 1$, while the underlying distribution is logistic. In this case, $e^*_{K,M}$ reduces to $15/2\pi^2 =$

0.7599, so that the Cramér-von Mises type test appears to have an edge over the Kolmogorov-Smirnov type test. For this example, $e^*_{M,T} = 2\pi^2/15 = 1.3159$, so that the terminal test is also not as efficient as the Cramér-von Mises type test. An opposite picture holds when one uses the Wilcoxon scores (viz., $\phi(u) = \sqrt{12}\,(u-1/2),\ 0 \le u \le 1$) while the underlying d.f. is exponential - here $e^*_{M,T} = 2\pi^2/35 \approx 0.5640$ and $e^*_{K,M} = 35/2\pi^2 = 1.2665$. These examples suggest that whereas the BARE $e^*_{M,T}$ or $e^*_{K,M}$ may fluctuate quite a bit for different score functions and underlying d.f.'s, $e^*_{K,T} \ge 1$ remains true under quite general conditions, tending to advocate the use of Komogorov-Smirnov type of PCS tests.

7. SIMULATED PERCENTILES OF NULL DISTRIBUTIONS $M^*_{N,r}$ AND $K^*_{N,r}$

The distributions of $M^*_{N,r}$ and $K^*_{N,r}$ have been shown to converge weakly to some functionals of the standard Wiener process under the null hypothesis and to those of the drifted Wiener process under contiguous alternatives, under certain regularity conditions. As we have mentioned in Section 4, the null distributions of these processes are not available in workable form. We, therefore, derive in this section a few percentile values of these distributions empirically through simulation studies.

Consider n independent observations, $Y_1,\dots,Y_n$ from the standard normal distribution. Let

$$S_k = \sum_{i=1}^{k} Y_k\ , \quad 1 \le k \le n\ ,$$

$$S_0 = 0 \ \text{by convention}\ . \tag{7.1}$$

We define the stochastic process $W_n = \{W_n(t),\ t \in [0,1]\}$ by letting

$$W_n(t) = n^{-\frac{1}{2}} S_{n(t)} , \tag{7.2}$$

where $n(t) = \max\{k\colon k \le tn\}$.

We note that the sample process W_n is right continuous with the left-hand limit and hence belongs to the metric space $D[0,1]$ with the properties

$$\begin{aligned} &EW_n(t) = 0 , \\ &EW_n^2(t) = n^{-1}[nt] , \\ &EW_n(t)W_n(t') = n^{-1}[n(t\wedge t')],\ t,t' \in [0,1] , \end{aligned} \tag{7.3}$$

where $[nt]$ and $[n(t\wedge t')]$ denote integral parts of nt and $n(t\wedge t')$ respectively. The maximum jump of the process is given by

$$\max_{1\le k\le n} \frac{|X_k|}{\sqrt{n}} = \frac{0(\sqrt{\ell_n \ell_n n})}{\sqrt{n}} \quad \text{a.s.}$$

$$\to 0 \quad \text{a.s., as} \quad n \to \infty . \tag{7.4}$$

Consequently, as n gets large the process W_n has a continuous sample path a.s. and has the structure of the standard Brownian motion process $W = \{W(t),\ t \in [0,1]\}$.

Let now $[\{Y_{ij}\}_{j=1}^{n},\ i = 1,\ldots,p]$ be independent random samples each of size n from $N(0,1)$. Then by Donsker's theorem [Billingsley (pp. 68-77; 1968)]

$$\max_{0\le k\le n}\left[\sum_{i=1}^{p} W_{in}^2(k/n)\right]^{\frac{1}{2}} \xrightarrow{\mathcal{D}} \omega_p^* ,$$

$$n^{-1}\sum_{i=1}^{p}\sum_{k=0}^{n-1} W_{in}^2(k/n) \xrightarrow{\mathcal{D}} \omega_p^0 , \qquad (7.5)$$

where $W_{in}(k/n) = n^{-\frac{1}{2}}\sum_{j=1}^{k} Y_{ij}$, $k = 0,\ldots,n$ and $i = 1,\ldots,p$ and ω_p^* and ω_p^0 are given by (4.28) and (4.29) respectively.

For purposes of the simulation studies, we have generated the standard normal deviates by using IBM scientific subroutine GAUSS. The sample size n has been taken to be 1000 and the empiricall (null) distributions of the two processes have been derived through 1000 independent repetitions. For details, see Majumdar (1976). In Table 1 and Table 2 below, we have furnished a few simulated values of the right tails of the two distributions. For $p = 1$, we have given exact percentile values of ω_p^* by using the approximation

$$P\left\{\sup_t |W(t)| \ge x\right\} \doteq 4(1-\Phi(x)) , \qquad (7.6)$$

where $\Phi(x)$ is the probability integral of the standard normal distribution.

TABLE 1: Simulated values of the null distribution of ω_p^* for selected values of p and α

α \ p	1*	2	3	4
.01	2.81	3.22	3.71	3.89
.05	2.24	2.70	3.05	3.31
.10	1.96	2.35	2.78	3.04

*Exact

TABLE 2: Simulated values of the null distribution of ω_p^0 for selected values of p and α

α \ p	1	2	3	4
.01	2.87	4.06	5.57	6.54
.05	1.66	2.67	3.49	4.33
.10	1.19	2.01	2.80	3.64

8. SOME GENERAL REMARKS

As has been mentioned in Section 1, our model (2.1) includes the multi-sample location model as a special case. Let X_{ji}, $1 \le i \le n_j$ be i.i.d.r.v. with a continuous d.f. $F_j(x)$, for $0 \le j \le p(\ge 1)$ and let $N = n_0 + \ldots + n_p$. Rewriting $X_{0i} = X_i$, $1 \le i \le n_0$ and $X_{ij} = X_{n_0+\ldots+n_{j-1}+i}$, $1 \le i \le n_j$, $1 \le j \le p$, and assuming the conventional location model where $F_j(x) = F(x-\theta_j)$, $0 \le j \le p$, we observe that (2.1) holds with $\beta_j = \theta_j - \theta_0$, $1 \le j \le p, \beta_0 = \theta_0$ and $\underset{\sim}{c}_1 = \ldots = \underset{\sim}{c}_{n_0} = \underset{\sim}{0}$, $\underset{\sim}{c}_{n_0+1} = \ldots = \underset{\sim}{c}_{n_0+n_1} = (1,0,\ldots,0)'$, $\ldots, \underset{\sim}{c}_{n_0+\ldots+n_{p-1}+1} = \ldots = \underset{\sim}{c}_N = (0,\ldots,0,1)'$. The null hypothesis H_0 in (2.2) insures that $F_0 = \ldots = F_p$. If, we assume that the sample sizes $n_0,\ldots,n_p$ satisfy the conditions

$$\lim_{N\to\infty} N^{-1} n_j = \lambda_j : 0 < \lambda_j < 1 \ , \quad \forall \ 0 \le j \le p \ , \tag{8.1}$$

then $\bar{\underset{\sim}{c}}_N \to (\lambda_0,\ldots,\lambda_p)'$ as $N \to \infty$ and, by (2.15), $N^{-1}\underset{\sim}{C}_N \to ((\delta_{kq}\lambda_k - \lambda_k\lambda_q))_{k,q=1,\ldots,p}$, so that (2.16) holds. Thus, the proposed PCS rank tests apply to the multi-sample location problem as well.

In order to test homogeneity of $k(=p+1)$ samples for right-censored data (fixed-plan censoring), with the smallest r out of N observations of the combined sample being considered, Basu (1967) has studied a generalized version of the Kruskal-Wallis test. The asymptotic chi-square distribution of his statistic (eqivalent to our $L^2_{N,r}/A^2_{N,r}$ for his particular model) follows readily from our Theorems 4.1 and 5.1. In his scheme, early termination of experimentation (prior to the r-th order statistic) has not been advocated, while in our PCS procedures, this is no problem. One can use $K^*_{N,r}$ or $M^*_{N,r}$. The BARE results of Section 6 suggests that using $K^*_{N,r}$ instead of $L^2_{N,r}/A^2_{N,r}$ allows an early termination without any loss of the asymptotic efficiency.

Suppose now that instead of a preassigned number r of failures, the experiment is designed to continue at most for a period of y time-units. Then, $r(y)$, the number of failures occurring in the time-period y, is itself a (non-negative integer-valued) random variable. As in Section 2 of Chatterjee and Sen (1973), the distribution theory of $M^*_{N,r(y)}$ or $K^*_{N,r(y)}$ can be developed [under the null hypothesis H_0 in (2.2)] under a conditional setup, given $r(y) = r$. However, in practice, this conditional argument requires some knowledge on the distribution of $r(y)$ so that the stochastic limit of $N^{-1}r(y)$ is fairly known in advance of experimentation (as the same is needed to define $A^2_{N,r}$ for both the PCS tests). We may surmount this problem by working with an upper bound for $N^{-1}r(y)$ [allowing chance fluctuation], whenever feasible. In the Department of Biostatistics, University of North Carolina, Chapel Hill, a seven-year project on the effect of high cholesterol on the risk of heart attack is under study. Male patients (over the age 35) are randomly allocated to either the control or treatment groups and the survival patterns of the two groups are being progressively studied. From independent sources (viz., U.S. Life tables), the seven year mortality rate for the particular age-pattern is roughly known to be about 11%, so that for a sample of size N, an upper (95% or 99%) confidence limit can be set on the actual number of failures in this study period, and with that upper limit, we can set our proposed PCS tests. This procedure, though a bit conservative, performs quite well (in scope as well as in performance) as compared to some parametric tests based on particular forms of failure distributions.

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For continuous observations from time-sequential studies, suitable Cramer-von Mises and Kolmogorov-Smirnov type (nonparametric) statistics (based on linear rank statistics) for testing hypotheses on some multiple regression models are proposed and studied. Asymptotic theory of these tests is provided for both the null and (local) alternative hypotheses situations and is based on the weak convergence of suitable rank order processes (on the D[0,1] space) to certain functions of Brownian motions. Bahadur efficiency results are also presented. Empirical values of the percentile points of

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20 Abstract

the null distributions of the proposed test statistics, obtained through simulation studies, are also provided.